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SOME EXOTIC ACTIONS OF FINITE GROUPS ON SMOOTH 4-MANIFOLDS

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Abstract

Using G -monopole invariants, we produce infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite number of $S^2 \times S^2$, $\mathbb{C}P_2$, $\overline{\mathbb{C}P}_2$, and $K3$ surfaces, where $k \geq 2$, and H is any nontrivial finite group acting freely on S^3 .

1. Introduction

The purpose of this paper is to present exotic, i.e. C^0 -equivalent but smoothly inequivalent smooth actions of finite groups on some smooth 4-manifolds. We say that two smooth group actions G_1 and G_2 on a smooth manifold M is C^m -equivalent for $m = 0, 1, \dots, \infty$, if there exists a C^m -homeomorphism $f: M \rightarrow M$ such that

$$G_1 = f \circ G_2 \circ f^{-1}.$$

Such exotic smooth actions of finite groups on smooth 4-manifolds have been found abundantly, for e.g., [7, 4, 9, 2, 10, 22, 8]. We showed that for any nontrivial finite group G there exists a smooth closed 4-manifold with infinitely many free G -actions which are all C^0 -equivalent but mutually smoothly inequivalent. And Fintushel, Stern, and Sunukjian constructed infinite families of exotic actions of finite cyclic groups on smooth closed 4-manifolds with nontrivial Seiberg–Witten invariant. All these examples are either free or cyclic actions.

In this article we use G -monopole invariants to detect infinitely many non-free non-cyclic exotic group actions on certain connected sums of 4-manifolds with vanishing Seiberg–Witten invariant. For example, for $k \geq 2$ and any nontrivial finite group H acting freely on S^3 , there exist infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite numbers of $S^2 \times S^2$, $\mathbb{C}P_2$, $\overline{\mathbb{C}P}_2$, and $K3$ surfaces.

2. Preliminaries on G -monopole invariant

Let M be a smooth closed oriented 4-manifold. Suppose that a finite group G acts on M smoothly preserving the orientation, and this action lifts to an action on a Spin^c structure \mathfrak{s} of M . For a G -invariant Riemannian metric and G -invariant perturbation ε ,

we consider a G -monopole moduli space \mathfrak{X} defined as the set of G -invariant solutions (A, Φ) of (perturbed) Seiberg–Witten equations

$$D_A \Phi = 0, \quad F_A^+ = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id} + \varepsilon$$

modulo the group $\mathcal{G}^G = \text{Map}(M, S^1)^G$ of G -invariant gauge transformations. As shown in [19, 20], \mathfrak{X} for a generic ε is a smooth compact orientable finite-dimensional manifold, if the dimension $b_2^+(M)^G$ of the space of G -invariant self-dual harmonic 2-forms on M is bigger than 0. In fact, it is a subset of the ordinary Seiberg–Witten moduli space.

The intersection theory on \mathfrak{X} using the universal cohomology classes of the ordinary Seiberg–Witten moduli space gives various G -monopole invariants defined first by Y. Ruan [17]. Considering gauge equivalence classes of G -invariant solutions under a based G -invariant gauge transformation group $\mathcal{G}_o^G = \{g \in \mathcal{G}^G \mid g(o) = 1\}$ for a fixed base point $o \in M$, we get a based G -monopole moduli space which is the principal S^1 -bundle over \mathfrak{X} induced by $\mathcal{G}^G/\mathcal{G}_o^G$ action. Let's denote its first Chern class by μ , which is independent of choice of the base point by the connectedness of M . We define a G -monopole invariant $SW_{M,\mathfrak{s}}^G$ as $\langle \mu^{(\dim \mathfrak{X})/2}, [\mathfrak{X}] \rangle$. (When $\dim \mathfrak{X}$ is odd, $SW_{M,\mathfrak{s}}^G$ is just set to be 0.)

As in the ordinary case, $SW_{M,\mathfrak{s}}^G$ is independent of the choice of a G -invariant metric and a G -invariant perturbation ε , if $b_2^+(M)^G > 1$. Thus we get a (smooth) topological invariant of a G -manifold M generalizing the ordinary Seiberg–Witten invariant $SW_{M,\mathfrak{s}}$, which is now $SW_{M,\mathfrak{s}}^{\{1\}}$ for the trivial group $\{1\}$. Also generalizing the Seiberg–Witten polynomial SW_M of M , the G -monopole polynomial of M is defined as

$$W_M^G SW_M^G := \sum_{\mathfrak{s}} SW_{M,\mathfrak{s}}^G PD(c_1(\mathfrak{s})) \in \mathbb{Z}[H_2(M; \mathbb{Z})^G],$$

where the summation is over the set of all G -equivariant Spin^c structures. Note that G -monopole invariants may change when a homotopically different lift of the G -action to the Spin^c structure is chosen. In a previous paper, we computed some examples of G -monopole invariants, which will be used as an essential tool in this paper:

Theorem 2.1 ([20]). *Let M and N be smooth closed oriented connected 4-manifolds satisfying $b_2^+(M) > 1$ and $b_2^+(N) = 0$, and \bar{M}_k for any $k \geq 2$ be the connected sum $M \# \cdots \# M \# N$ where there are k summands of M .*

Suppose that a finite group G with $|G| = k$ acts effectively on N in a smooth orientation-preserving way such that it is free or has at least one fixed point, and that N admits a Riemannian metric of positive scalar curvature invariant under the G -action and a G -equivariant Spin^c structure \mathfrak{s}_N with $c_1^2(\mathfrak{s}_N) = -b_2(N)$.

Define a G -action on \bar{M}_k induced from that of N permuting the k summands of M glued along a free orbit in N , and let $\bar{\mathfrak{s}}$ be the Spin^c structure on \bar{M}_k obtained by

gluing \mathfrak{s}_N and a Spin^c structure \mathfrak{s} of M .

Then for any G -action on $\bar{\mathfrak{s}}$ covering the above G -action on \bar{M}_k ,

$$SW_{\bar{M}_k, \bar{\mathfrak{s}}}^G \equiv SW_{M, \mathfrak{s}} \pmod{2},$$

if the dimension $b_1(N)^G$ of the vector space consisting of G -invariant elements of $H_1(N; \mathbb{R})$ is zero.

Note that if a smooth closed manifold X has a smooth effective action of a compact Lie group G , then the fixed-point set X^g under $g \in G$ is either empty or an embedded submanifold each component of which has positive codimension. Thus N in the above theorem always has a free orbit under G . When $b_1(N)^G \neq 0$, we also obtained a mod 2 equality relating those two invariants, but we omit it here for simplicity. The examples of such N with $G = \mathbb{Z}_k$ regardless of $b_1(N)^{\mathbb{Z}_k}$ are as follows:

Theorem 2.2 ([20]). *Let X be one of*

$$S^4, \quad \overline{\mathbb{C}P}_2, \quad S^1 \times (L_1 \# \cdots \# L_n), \quad \text{and} \quad \widehat{S^1 \times L}$$

where each L_i and L are quotients of S^3 by free actions of finite groups, and $\widehat{S^1 \times L}$ is the manifold obtained from the surgery on $S^1 \times L$ along an $S^1 \times \{pt\}$.

Then for any integer $l \geq 0$ and any smooth closed oriented 4-manifold Z with $b_2^+(Z) = 0$ admitting a metric of positive scalar curvature,

$$X \# klZ$$

satisfies the properties of N in Theorem 2.1 with $G = \mathbb{Z}_k$, where the Spin^c structure of $X \# klZ$ is given by gluing any Spin^c structure \mathfrak{s}_X on X and any Spin^c structure \mathfrak{s}_Z on Z satisfying $c_1^2(\mathfrak{s}_X) = -b_2(X)$ and $c_1^2(\mathfrak{s}_Z) = -b_2(Z)$ respectively.

3. Exotic group actions

Following [12], we say that a simply connected 4-manifold *dissolves* if it is diffeomorphic to either

$$n\mathbb{C}P_2 \# m\overline{\mathbb{C}P}_2$$

or

$$\pm(n(S^2 \times S^2) \# mK3)$$

for some $n, m \geq 0$ according to its homeomorphism type. We also use the term mod 2 basic class to mean the first Chern class of a Spin^c structure with nonzero mod 2 Seiberg–Witten invariant.

Theorem 3.1. *Let M be a smooth closed oriented connected 4-manifold and $\{M_i \mid i \in \mathfrak{I}\}$ be a family of smooth 4-manifolds such that every M_i is homeomorphic to M and the numbers of mod 2 basic classes of M_i 's are all mutually different, but each $M_i \# l_i(S^2 \times S^2)$ is diffeomorphic to $M \# l_i(S^2 \times S^2)$ for an integer $l_i \geq 1$.*

If $l_{\max} := \sup_{i \in \mathfrak{I}} l_i < \infty$, then for any integers $k \geq 2$ and $l \geq l_{\max} + 1$,

$$klM \# (l-1)(S^2 \times S^2)$$

admits an \mathfrak{I} -family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ where H is any group of order l acting freely on S^3 .

Proof. Think of $klM \# (l-1)(S^2 \times S^2)$ as

$$klM_i \# (l-1)(S^2 \times S^2),$$

and our H action is defined as the deck transformation map of the l -fold covering map onto

$$\bar{M}_{i,k} := kM_i \# \widehat{S^1 \times L}$$

where $\widehat{S^1 \times L}$ for $L = S^3/H$ is defined as in Theorem 2.2. To define a \mathbb{Z}_k -action, note that $\bar{M}_{i,k}$ has a \mathbb{Z}_k -action coming from the \mathbb{Z}_k -action of $\widehat{S^1 \times L}$ defined in Theorem 2.2, which is basically a rotation along the S^1 -direction. This \mathbb{Z}_k action is obviously lifted to the above l -fold cover, and it commutes with the above defined H action. Thus we have an \mathfrak{I} -family of $\mathbb{Z}_k \oplus H$ actions on $klM \# (l-1)(S^2 \times S^2)$, which are all topologically equivalent by using the homeomorphism between each M_i and M .

Recall from Theorem 2.2 and its proof in [20] that all the Spin^c structures on a spin manifold $\widehat{S^1 \times L}$ are \mathbb{Z}_k -equivariant with $c_1^2 = -b_2(\widehat{S^1 \times L}) = 0$, and hence \mathbb{Z}_k -equivariant Spin^c structures on $\bar{M}_{i,k}$ are parametrized by

$$H_2(\bar{M}_{i,k}; \mathbb{Z})^{\mathbb{Z}_k} \cong H_2(M_i; \mathbb{Z}) \oplus H_2(\widehat{S^1 \times L}; \mathbb{Z}).$$

By Theorem 2.1 and the fact that $b_1(\widehat{S^1 \times L}) = 0$, for any Spin^c structure \mathfrak{s}_i on M_i ,

$$SW_{\bar{M}_{i,k}, \bar{\mathfrak{s}}_i}^{\mathbb{Z}_k} \equiv SW_{M_i, \mathfrak{s}_i} \pmod{2},$$

and hence

$$SW_{\bar{M}_{i,k}}^{\mathbb{Z}_k} \equiv SW_{M_i} \sum_{[\alpha] \in H_2(\widehat{S^1 \times L}; \mathbb{Z})} [\alpha]$$

modulo 2. Therefore $SW_{\bar{M}_{i,k}}^{\mathbb{Z}_k} \pmod{2}$ for all i have mutually different numbers of monomials, and hence the \mathfrak{I} -family of $\mathbb{Z}_k \oplus H$ actions on $klM \# (l-1)(S^2 \times S^2)$ cannot be smoothly equivalent, completing the proof. \square

Corollary 3.2. *Let H be a finite group of order $l \geq 2$ acting freely on S^3 . For any $k \geq 2$, there exists an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ on*

$$\begin{aligned} & (klm + l - 1)(S^2 \times S^2), \\ & (kl(n - 1) + l - 1)(S^2 \times S^2) \# klnK3, \\ & (kl(2n' - 1) + l - 1)\mathbb{C}P_2 \# (kl(10n' + m' - 1) + l - 1)\overline{\mathbb{C}P}_2 \end{aligned}$$

for infinitely many m , and any $m' \geq 1$, $n, n' \geq 2$.

Proof. By the result of B. Hanke, D. Kotschick, and J. Wehrheim [13], $m(S^2 \times S^2)$ for infinitely many m has the property of M in the above theorem with each $l_i = 1$ and $|\mathcal{J}| = \infty$. The different smooth structures of their examples are constructed by fiber-summing a logarithmic transform of $E(2n)$ and a certain symplectic 4-manifold along a symplectically embedded torus, and different numbers of mod 2 basic classes are due to those different logarithmic transformations. Indeed the Seiberg–Witten polynomial of the multiplicity r logarithmic transform of $E(2n)$ is given by

$$([T]^r - [T]^{-r})^{2n-2}([T]^{r-1} + [T]^{r-3} + \cdots + [T]^{1-r})$$

whose number of terms with coefficients mod 2 can be made arbitrarily large by taking r sufficiently large, and the fiber sum with the other symplectic 4-manifold is performed on a fiber in an $N(2)$ disjoint from the Gompf nucleus $N(2n)$ where the log transform is performed so that all these mod 2 basic classes survive the fiber-summing by the gluing formula of C. Taubes [21]. Therefore $(klm + l - 1)(S^2 \times S^2)$ has desired actions by the above theorem.

For the second example, we use a well-known fact that $E(n)$ for $n \geq 2$ also has the above properties of M in the above theorem with each $l_i = 1$, where its exotica M_i 's are $E(n)_K$ for a knot $K \subset S^3$ by the Fintushel–Stern knot surgery. Recall the theorem by S. Akbulut [1] and D. Auckly [3] which says that for any smooth closed simply-connected X with an embedded torus T such that $T \cdot T = 0$ and $\pi_1(X - T) = 0$, a knot-surgered manifold X_K along T via a knot K satisfies

$$X_K \# (S^2 \times S^2) = X \# (S^2 \times S^2).$$

And from the formula

$$SW_{E(n)_K} = \Delta_K([T]^2)([T] - [T]^{-1})^{n-2}$$

where Δ_K is the symmetrized Alexander polynomial of K , one can easily see that the number of mod 2 basic classes of $E(n)_K$ can be made arbitrarily large by choosing K

appropriately. (For example, take K with

$$\Delta_K(t) = 1 + \sum_{j=1}^{2d} (-1)^j (t^{jn} + t^{-jn})$$

for sufficiently large d .) Therefore

$$klE(2n) \# (l-1)(S^2 \times S^2) = kl n K3 \# (kl(n-1) + l-1)S^2 \times S^2$$

has desired actions, where we used the fact that $S \# (S^2 \times S^2)$ dissolves for any smooth closed simply-connected elliptic surface S by the work of R. Mandelbaum [14] and R. Gompf [11].

For the third example, one can take M to be $E(n') \# m' \overline{\mathbb{C}P}_2$ for $n' \geq 2$, $m' \geq 1$, where its exotica M_i 's are $E(n')_K \# m' \overline{\mathbb{C}P}_2$ for a knot $K \subset S^3$, because

$$\begin{aligned} SW_{E(n')_K \# m' \overline{\mathbb{C}P}_2} &= SW_{(E(n') \# m' \overline{\mathbb{C}P}_2)_K} \\ &= \Delta_K([T]^2)([T] - [T]^{-1})^{n'-2} \prod_{i=1}^{m'} ([E_i] + [E_i]^{-1}), \end{aligned}$$

where E_i 's denote the exceptional divisors, and we used the fact that $E(n')$ is of simple type. Since $E(n') \# \overline{\mathbb{C}P}_2$ for any n' is non-spin,

$$kl(E(n') \# m' \overline{\mathbb{C}P}_2) \# (l-1)(S^2 \times S^2) = kl(E(n') \# m' \overline{\mathbb{C}P}_2) \# (l-1)(\mathbb{C}P_2 \# \overline{\mathbb{C}P}_2),$$

and it dissolves into the connected sum of $\mathbb{C}P_2$'s and $\overline{\mathbb{C}P}_2$'s, using the dissolution ([14, 11]) of $E(n') \# \mathbb{C}P_2$ into $2n' \mathbb{C}P_2 \# (10n' - 1) \overline{\mathbb{C}P}_2$. \square

REMARK. For other combinations of $K3$ surfaces and $(S^2 \times S^2)$'s in the above corollary, one can use B. Hanke, D. Kotschick, and J. Wehrheim's other examples in [13]. One can also construct many other such examples of M with infinitely many exotica which become diffeomorphic after one stabilization by using the knot surgery.

Any finite group acting freely on S^3 is in fact a subgroup of $SO(4)$ by the well-known result of G. Perelman ([15, 16]), and Theorem 3.1 and Corollary 3.2 can be generalized a little further. (See [18].)

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